ALGEMEEN (4) Ken so goed as moontlik vir deelname aan kompetisies.
GENERAL (4) Know as well as possible for participation in competitions.

ALGEMEEN 4 / GENERAL 4 Gr 7 - 9

PERMUTASIES, KOMBINASIES EN WAARSKYNLIKHEID — INLEIDEND

PERMUTATIONS, COMBINATIONS AND PROBABILITY — INTRODUCTORY

$0! = 1$

1. Kombinasies / Combinations

Volgorde is nie belangrik nie / Order is not important

Notation / Notasie $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

(Spreek uit as *n* kombinasie *r* / Pronounce as *n* combination *r*)

Die getal muntstukke waaruit ons kan kies, is bv. *n* en die getal wat ons wel kies, is r.

(a) Kies al drie die letters uit *a*, *b* en *c*. / Choose all three letters from *a*, *b* and *c*.
$n = 3, \ r = 3$

$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = \frac{3\times2\times1}{3\times2\times1(0)!} = \frac{1}{0!} = \frac{1}{1} = 1$$

1 kombinasie/combination

(b) Kies twee uit die drie letters *a*, *b*, en *c*: / Choose two of the three letters *a*, *b* and *c*:
$n = 3, r = 2$

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3\times2\times1}{2\times1(1)!} = \frac{3}{1} = 3$$

ab, *ac*, *bc* $\Rightarrow$ 3 kombinasies/combinations
ab is dieselfde as *ba*, *ac* is dieselfde as *ca* en *bc* is dieselfde as *cb*.
ab is the same as *ba*, *ac* is the same as *ca* en *bc* is the same as *cb*.

(c) Kies twee uit die vier letters *a*, *b*, *c* en *d*: / Choose two of the four letters *a*, *b*, *c* and *d*:
$n = 4, r = 2$

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4\times3\times2\times1}{2\times1(2)!} = \frac{12}{2} = 6$$

ab, ac, ad, bc, bd, cd $\Rightarrow$ 6 kombinasies/combinations

(d) As 'n klub uit 20 lede bestaan en 'n komitee van vier lede word gekies, is volgorde nie belangrik nie en word kombinasies gebruik. $n = 20$ en $r = 4$.

Dus is $\binom{20}{4} = \frac{20!}{4!(20-4)!} = \frac{20!}{16!\times4!} = \frac{20\times19\times18\times17}{4\times3\times2\times1} = 4845$

Die komitee kan dus op 4 845 maniere gekies word.

If a club has 20 members and a committee of four members must be chosen, order is not important and combinations are used.

Hence $\binom{20}{4} = \frac{20!}{4!(20-4)!} = \frac{20!}{16! \times 4!} = \frac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2 \times 1} = 4845$

The committee can be chosen in 4 845 ways.

(e) Let wel: / Note:

★$n! = n(n-1)!$

$6! = 6(6-1)! = 6 \cdot 5!$

★$(n-r)! = (n-r)(n-r-1)!$

$LK = (7-4)! = 3!$

$RK = (7-4)(7-4-1)! = 3 \cdot 2! = 3!$

$\therefore$ LK = RK

★$(n-r-1)! = \frac{(n-r)!}{(n-r)}$

$LK = (7-4-1)! = 2!$

$RK = \frac{(7-4)!}{(7-4)} = \frac{3!}{3} = \frac{3 \cdot 2!}{3} = 2!$

$\therefore$ LK = RK

2. Permutasies / Permutations

Volgorde is belangrik / Order is important

Notation / Notasie $P(n,r) = \frac{n!}{(n-r)!}$

(Spreek uit as *n* permutasie *r* / Pronounce as *n* permutation *r*)

As al *n* elemente gebruik word, is die getal rangskikkings $n!$, want $n = r$ en $(n-r)! = 0! = 1$. Die eerste element kan op *n* maniere gekies word, die tweede op $(n-1)$ maniere, . . . en die laaste een op 1 manier.

If all *n* elements are used, the number of arrangements is $n!$, since $n = r$ and $(n-r)! = 0! = 1$. The first can be chosen in *n* ways, the second in $(n-1)$ ways, . . . and the last in 1 way.

Die getal permutasies van *n* items met *m* wat identies is $(m < n)$, is $\frac{n!}{m!}$.

The number of permutations of *n* items with *m* identical items is $(m < n)$, is $\frac{n!}{m!}$.

(a) Die permutasie van die letters *a*, *b* en *c* as al drie letters gekies word: / The permutation of *a*, *b* and *c* if all three letters are chosen:
$n = 3;\ r = 3$

$P(3,3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{6}{1} = 6$

abc, *acb*, *bac*, *bca*, *cab*, *cba* $\Rightarrow$ 6 permutasies/permutations

(b) Kies twee letters uit die drie letters *a*, *b* en *c*.
Choose two letters out of the three letters *a*, *b* and *c*:
$n = 3;\ r = 2$

$$P(3,2) = \frac{3!}{(3-2)!} = \frac{3!}{1!} = \frac{6}{1} = 6$$

ab, *ba*, *ac*, *ca*, *bc*, *cb* $\Rightarrow$ 6 permutasies/permutations

(c) Kies twee letters uit die vier letters *a*, *b*, *c* en *d*.
Choose two letters out of the four letters *a*, *b*, *c* and *d*.
$n = 4$; $r = 2$

$$P(4,2) = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4 \times 3 \times 2!}{2!} = 12$$

ab, *ba*, *ac*, *ca*, *ad*, *da*, *bc*, *cb*, *bd*, *db*, *cd*, *dc* $\Rightarrow$ 12 permutasies/permutations

(d) As 'n klub uit 20 lede bestaan, en 'n president, vise-president, sekretaris en tesourier moet gekies word, is volgorde belangrik en word permutasies gebruik.

Dus kan hierdie komitee op $P(20,4) = \frac{20!}{(20-4)!} = \frac{20!}{16!} = 20 \times 19 \times 18 \times 17 = 116\,280$ maniere gekies word.

If a club has 20 members, and a president, vice-president, secretary and treasurer must be chosen, the order is important and permutations are used.
Therefore this committee can be chosen in

$P(20,4) = \frac{20!}{(20-4)!} = \frac{20!}{16!} = 20 \times 19 \times 18 \times 17 = 116\,280$ ways.

(e) Hoeveel rangskikkings is daar vir die letters in die woord "BALL"?
How many arrangements are there for the letters in the word "BALL"?

Die L herhaal, dus is daar $\frac{24}{2!}$ = 12 rangskikkings.

The L repeats, and there are $\frac{24}{2!}$ = 12 arrangements.

(f) "TATTER": $\frac{6!}{(6-6)!3!} = \frac{6!}{3!}$ = 120 rangskikkings/arrangements. (3 T's)

"PAPA": $\frac{4!}{(4-4)!2!2!} = \frac{4!}{2!\,2!}$ = 6 rangskikkings/arrangements. (2 P's & 2 A's)

"BANANA": $\frac{6!}{(6-6)!2!3!} = \frac{6!}{2!\,3!}$ = 60 rangskikkings/arrangements. (3 A's & 2 N's)

3.1 Pascal se Driehoek / Pascal's Triangle

Dit is baie maklik om te onthou hoe die driehoek lyk en hoe om die waardes te kry.
It is very easy tot remember what the triangle looks like and how to obtain the values.

ALGEMEEN (4) Ken so goed as moontlik vir deelname aan kompetisies.
GENERAL (4) Know as well as possible for participation in competitions.

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$\rightarrow$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Kombinasies / Combinations: $\frac{n!}{r!(n-r)!} = \frac{4!}{3!(4-3)!} = 4$ en $\frac{n!}{r!(n-r)!} = \frac{4!}{1!(4-1)!} = 4$

Die 4 dui op ry 4; tel die rye van 0 af. Die 3 dui op element 3 van links af; tel van 0 af.
Die 4 dui op ry 4; tel die rye van 0 af. Die 1 dui op element 1 van links af; tel van 0 af.

The 4 indicates row 4; count the rows from 0. The 3 indicates element 3 from the left; count from 0.
The 4 indicates row 4; count the rows from 0. The 1 indicates element 1 from the left; count from 0.

Later sal julle die binomiaalstelling leer wat hiermee te doen het.
Later you will learn the *binomial theorem* from which we get these values.

Wanneer hakies soos die volgende uitgeskryf word, kan ons Pascal se Driehoek gebruik om die koëffisiënte direk neer te skryf.

When brackets such as the following are written out we can use Pascal's Triangle to write down the coefficients directly.

$(a + b)^0 = \mathbf{1}a^0b^0 = 1$ . . row 0

$(a + b)^1 = \mathbf{1}a^1b^0 + \mathbf{1}a^0b^1 = a + b$ . . row 1

$(a + b)^2 = \mathbf{1}a^2b^0 + \mathbf{2}a^1b^1 + \mathbf{1}a^0b^2 = a^2 + 2ab + b^2$ . . row 2

$(a + b)^3 = \mathbf{1}a^3b^0 + \mathbf{3}a^2b^1 + \mathbf{3}a^1b^2 + \mathbf{1}a^0b^3 = a^3 + 3a^2b + 3ab^2 + b^3$ . . row 3

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

$_4C_3 = 4$
ry 4, element 3
row 4, element 3

Tel die rye en elemente van 0 af. / Count the rows and elements from 0.

3.2. Voorbeelde / Examples:

1. $(3x - 1)^7$

$= 1(3x)^7(-1)^0 + 7(3x)^6(-1)^1 + 21(3x)^5(-1)^2 + 35(3x)^4(-1)^3 + 35(3x)^3(-1)^4$
$+ 21(3x)^2(-1)^5 + 7(3x)^1(-1)^6 + 1(3x)^0(-1)^7$

$= 2187x^7 - 5103x^6 + 5103x^5 - 2835x^4 + 945x^3 - 189x^2 + 21x - 1$

2. $(x + 3)^4$

$= 1x^4 3^0 + 4x^3 3^1 + 6x^2 3^2 + 4x^1 3^3 + 1x^0 3^4 = x^4 + 12x^3 + 54x^2 + 108x + 81$

3. $(x - 3y)^4 = 1x^4(-3y)^0 + 4x^3(-3y)^1 + 6x^2(-3y)^2 + 4x^1(-3y)^3 + 1x^0(-3y)^4$

$= x^4 - 12x^3y + 54x^2y2 - 108xy^3 + 81y^4$

4. Kan jy die driehoekgetalle in Pascal se Driehoek identifiseer?

Can you identify the triangular numbers in Pascal's Triangle?

5.1A Hoeveelheid verskillende paaie

Die meeste gevalle waar die aantal verskillende paaie van 'n punt A na 'n punt B bereken moet word, werk in twee rigtings, bv. op en na regs. Soms word die rigtings aangedui deur noord/oos of suid/wes. Hierdie rigtings word bepaal deur die begin- en eindpunte van die paaie.

Die vraag sou ook kon wees om die lengte van die kortste pad tussen twee punte te bepaal.

As jy slegs die aantal verskillende paaie tussen twee punte wil bepaal, is dit nie nodig dat die vierkante of blokkies identies hoef te wees nie, maar as jy die kortste pad moet bepaal, moet die blokkies of vierkante identies wees.

Daar is twee maniere om hierdie tipe probleem op te los. Die ① een manier is om die blokkies te tel en die ② ander manier is om van kombinasies en permutasies gebruik te maak.

5.1E Number of different paths

Most cases where the number of different paths from a point A to a point B on a grid must be calculated, works in two directions, e.g. up and to the right. Sometimes directions are indicated by north/east or south/west. These directions will be determined by the start and end points of the paths.

The question can also be changed and can require you to determine the length of the shortest path between two points.

If you only want the number of different paths between two points, it is not necessary that squares or blocks have to be identical, but if the shortest path has to be determined the blocks or squares must be identical.

There are actually two ways to solve this kind of problem. The ① one method is to simply count the blocks while the ② other method makes use of combinations and permutations.

Let wel: / Note:

$$\binom{0}{0} = \frac{0!}{(0-0)!\,0!} = 1 = \mathbf{0!} \qquad \binom{1}{0} = \frac{1!}{(1-0)!\,0!} = 1 \qquad \binom{1}{1} = \frac{1!}{(1-1)!\,1!} = 1$$

★★ Voorbeelde / Examples

Let wel: Jy kan van een punt op 'n rooster na 'n ander punt op die rooster beweeg; of van een vierkant na 'n ander vierkant.
Note: You can move from one point to another point on the grid; or from one square to another square.

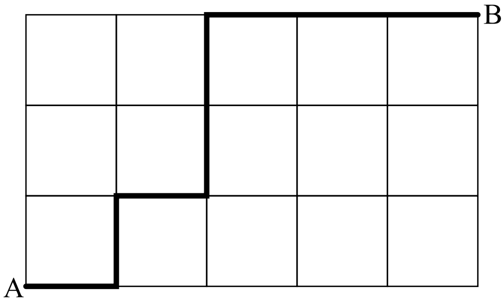
1.

♦ Bepaal hoeveel verskillende paaie daar is van punt A tot by punt B as jy net op en regs mag beweeg:

In hierdie geval bestaan elke pad uit 8 stappe: 3 na bo (**op** waarvoor ons die letter ***u*** gebruik) en 5 na regs (***r***), bv. die pad ***ruruurrr***.
Daar is 3 herhalings vir "op" en 5 herhalings vir "regs". Die aantal verskillende paaie van A na B is dus $\frac{(3+5)!}{5!\,3!} = \frac{8!}{5!\,3!} = \frac{8 \times 7 \times 6}{6} = 56$.

♦ Determine the number of different paths from point *A* to point *B* if you may only move up and to the right:
In this case each path consists of eight steps: 3 up-steps (***u***) and 5 right-steps (***r***), e.g. the path ***ruruurrr***.

There are 3 repetitions for "up" and 5 repetitions for "right". The number of different paths from A to B is therefore $\frac{(3+5)!}{5!\,3!} = \frac{8!}{5!\,3!} = \frac{8 \times 7 \times 6}{6} = 56$.



of/or

1	4	10	20	35	56
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1

Die getalle is op die hoekpunte omdat A en B op die hoekpunte is.
The numbers are on the vertices, since A and B are on the vertices.

2.

Die aantal verskillende paaie van A na die hoek onder regs word gedoen soos in die figuur van Pascal se driehoek. Die getalle is in die vierkante omdat A binne 'n vierkant is.

The number of all the diffferent paths from A to the lower right corner of the lattice, is done as shown in the figure if Pascal's Triangle is used (the numbers are inside the squares, since A is inside a square):

A	1	1	1	1	1
1	2	3	4	5	6
1	3	6	10	15	21
1	4	10	20	35	56
1	5	15	35	70	126
1	6	21	56	126	252

Ons gebruik *kombinasies* en $n = 5 + 5 = 10$ en $r = 5$ sodat $n - r = 5$. Laat n die aantal skuiwe na onder en na regs wees. Dus, 5 af en 5 na regs. En, r is die aantal skuiwe na regs en dit is 5. Dan is $n - r$ die aantal skuiwe na onder. Die totale aantal verskillende paaie is dan

$\frac{n!}{r!(n-r)!} = \frac{10!}{5!\,5!} = 252.$

Using *combinations* we take $n = 5 + 5 = 10$ and $r = 5$; then $n - r = 5$, where r can be the number of moves to the right. We let n be the number of moves downwards and to the right which is 5 + 5. So, $n - r$ will then be the number of down moves. The total number of different paths is then $\frac{n!}{r!(n-r)!} = \frac{10!}{5!\,5!} = 252.$

2.1A In Pascal se geval kan ons ook slegs permutasies gebruik. Die totale aantal skuiwe is dan 10 = n, en daar is twee soorte skuiwe. Daar is vyf herhalings van elke soort. Dus het ons (10!) / (5! 5!) = 252.

(Vergelyk die woord *mamma* wat uit 5 letters bestaan met 2 *a*'s en 3 *m*'e. Die aantal permutasies is dan (2 + 3)! / (2! 3!) = 5! / (2! 3!) = 10. Hier is twee soorte letters en dit stem ooreen met die twee rigtings van die skuiwe.)

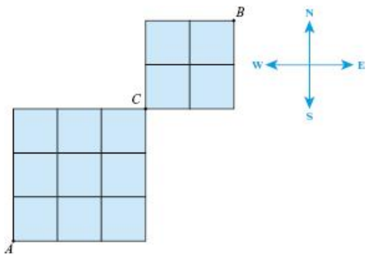
2.1E In Pascal's case one can also consider *permutations*. Total number of moves = 10 = n, and there are two kinds of moves. There are 5 repetitions of each kind. Then we have (10!) / (5! 5!) = 252.

(Compare to the word *mamma* that consists of 5 letters with 2 *a*'s and 3 *m*'s. The number of permutations is then (2 + 3)! / (2! 3!) = 5! / (2! 3!) = 10. Here are 2 kinds of letters and that correspond to the 2 directions of the moves.)

3.

As jy slegs noord (op) of oos (regs) kan beweeg, bepaal die aantal verskillende paaie van A na B in die volgende rangskikking van strate.

If you can travel only north (up) or east (right), determine the number of different paths from *A* to *B* in the following street arrangement.



3.1 Die aantal verskillende paaie van A na C: $\frac{(3+3)!}{3!\,3!} = 20$

Die aantal verskillende paaie van C na B: $\frac{(2+2)!}{2!\,2!} = 6$

Dus is die aantal verskillende paaie van A na B = 20 × 6 = 120.

3.1 The number of different paths from A to C: $\frac{(3+3)!}{3!\,3!} = 20$

The number of different paths from C to B: $\frac{(2+2)!}{2!\,2!} = 6$

Hence, the number of different paths from A to B: 20 × 6 = 120.

3.2 Gebruik Pascal se driehoek (die klein getalle by die hoekpunte)

3.2 Use Pascal's Triangle (the small numbers are at the vertices)

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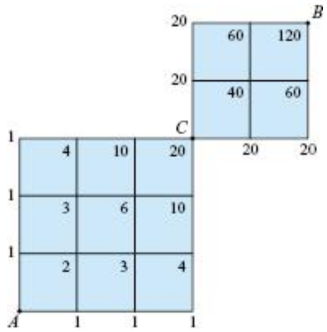
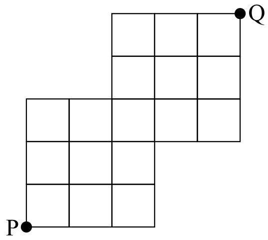
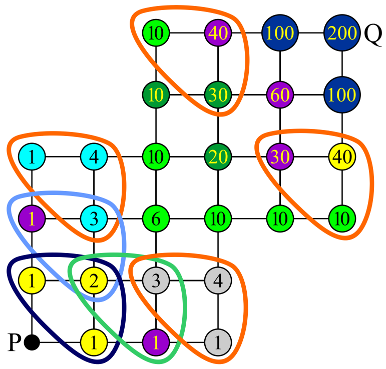


4. Hoeveel verskillende kortste paaie is daar van P na Q? (Hier is dit belangrik dat die vierkante identies moet wees.

How many different shortest paths are there from P to Q? (Here it is important that the squares must be identical.)



In elke driehoek word die waarde links bo by die waarde regs onder getel om die waarde regs bo te kry. Ons gebruik hier dus Pascal se driehoek.
In each *triangle* the upper left value is added to the lower right value to obtain the upper right value. This method involves Pascal's triangle.



5. Op die 6 × 6-bord is daar drie versperrings wat met dik lyne geteken is. 'n Mier is in die hoek links bo en wil na punt A gaan. Die mier mag slegs na 'n vierkant gaan wat 'n sy met die huidige vierkant deel en dit mag net af of na regs beweeg. Dit kan nie deur 'n versperring gaan nie en dit kan ook nie diagonaal (skuins) beweeg nie. Hoeveel verskillende paaie is daar waarop die mier kan beweeg?
On the 6 × 6-board are three barriers, drawn with thick lines. An ant is in the upper left corner and wants to go to point A. It may only move from one square to another square that shares a side, and it may only move down or to the right. It cannot go through a barrier and it cannot move diagonally. How many different paths are there that the ant can follow?

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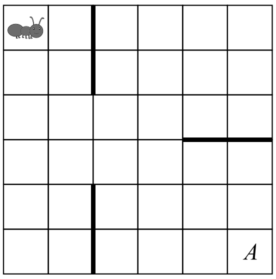
GENERAL (4) Know as well as possible for participation in competitions.



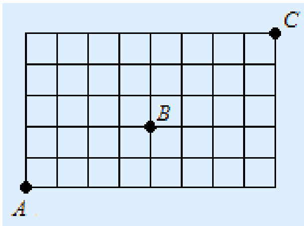
1	1	0	0	0	0
1	2	0	0	0	0
1	3	3	3	3	3
1	4	7	10	10	10
0	0	7	17	27	37
0	0	7	24	51	88

Die rooster aan die regterkant toon die aantal verskillende paaie (88) wat die mier kan volg om by punt A uit te kom as die versperrings in ag geneem word. Pascal se driehoek is die maklikste metode in hierdie geval.

The grid on the right shows the number of different paths (88) the ant can follow to point A when the barriers are taken into account. Pascal's triangle is the easiest method in this case.

6. 1 Hoeveel verskillende roetes is daar van *A* na *C* wat oor *B* gaan, sonder ompaaie?

How many different routes are there from *A* to *C* via *B*, without detours?

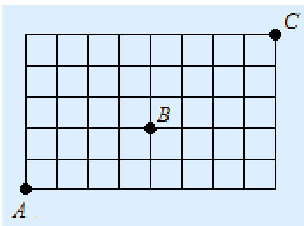


Permutasies:
From A to B there are (6!) ÷ (2! × 4!) = (6 × 5) ÷ (2) = 15 different paths
From B to C there are (7!) ÷ (3! × 4!) = (7 × 6 × 5) ÷ (3 × 2) = 35 different paths
Total number of different paths = 15 × 35 = 525 different paths.

				15	60	150	300	525 C
				15	45	90	150	225
				15	30	45	60	75
1	3	6	10	15	15	15	15	15
1	2	3	4	5				
1	1	1	1	1				

6. 2 Hoeveel verskillende roetes is daar van *A* na *C* wat nie deur *B* gaan nie?

How many different routes are there from *A* to *C* not passing through *B*?



ALGEMEEN (4) Ken so goed as moontlik vir deelname aan kompetisies.
GENERAL (4) Know as well as possible for participation in competitions.

1	6	21	56	111	192	312	492	762	C
1	5	15	35	55	81	120	180	270	
1	4	10	20	20	26	39	60	90	
1	3	6	10	0	6	13	21	30	
1	2	3	4	5	6	7	8	9	
1	1	1	1	1	1	1	1	1	

Daar is 525 verskillende paaie van *A* na *C* via *B*.
Daar is 762 verskillende paaie van *A* na *C* wat nie deur *B* gaan nie.
In totaal is daar 1287 verskillende paaie van *A* na *C* = 525 + 762.
525 ÷ 1287 = 0.41. Ongeveer 41% van die paaie gaan deur B.

There are 525 different paths from *A* to *C* via *B*.
There are 762 different paths from *A* to *C* not passing through *B*.
In total there are 1287 different paths from *A* to *C* = 525 + 762.
525 ÷ 1287 = 0.41. About 41% of the paths go through B.

7. In hierdie voorbeeld het ons 'n 6 × 6-rooster soos aangetoon. In voorbeeld 6 het ons van hoekpunt na hoekpunt beweeg. Nou gaan ons van vierkant na vierkant beweeg.

Take a 6 × 6 lattice as shown in the figure. In Example 6 we moved on the lines from vertex to vertex. In this example we will move from block to block.

A					
		B			
					C

7.1 Hoeveel verskillende roetes is daar van *A* na *C* wat oor *B* gaan, sonder ompaaie?

How many different routes are there from *A* to *C* via *B*, without detours?

0	1	1	0	0	0
1	2	3	0	0	0
1	3	6	6	6	6
0	0	6	12	18	24
0	0	6	18	36	60
0	0	6	24	60	120

Die oop spasies kan met nulle gevul word omdat geen veranderinge daar plaasvind nie.

The open spaces can be filled with zeros, since no changes take place there.

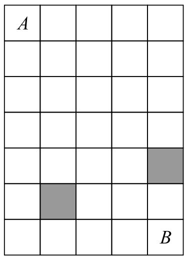
ALGEMEEN (4) Ken so goed as moontlik vir deelname aan kompetisies.
GENERAL (4) Know as well as possible for participation in competitions.

7.2 Hoeveel verskillende roetes is daar van *A* na *C* wat nie oor *B* gaan nie?

How many different routes are there from *A* to *C* not passing through *B*?

0	1	1	1	1	1
1	2	3	4	5	6
1	3	0	4	9	15
1	4	4	8	17	32
1	5	9	17	34	66
1	6	15	32	66	132

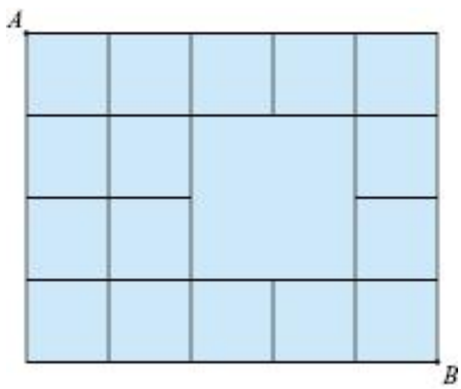
8. How many different paths are there that start at A, do not go through any of the shaded squares, and end at B?



Paaie / Paths:

0	1	1	1	1
1	2	3	4	5
1	3	4	5	6
1	4	5	6	7
1	5	10	16	0
1	0	10	26	26
1	1	11	37	63

9. How many different paths are there from A to B?



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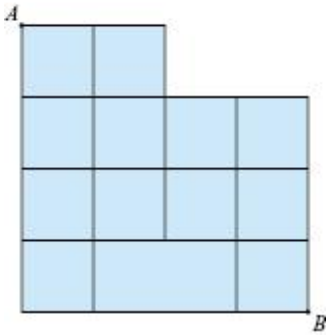
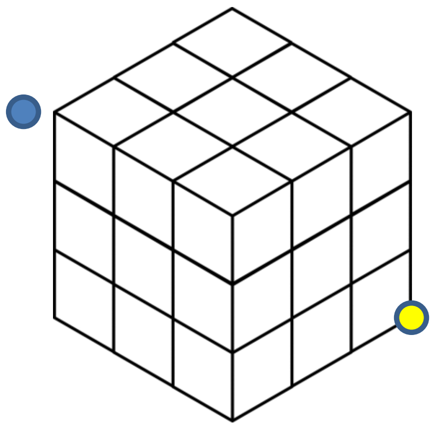
1	1	1	1	1	1
1	2	3	4	5	6
1	3	6	0	5	11
1	4	10	10	15	26
1	5	15	25	40	66

10. How many different paths are there from A to B?



1	1	1	0	0
1	2	3	3	3
1	3	6	9	12
1	4	10	19	31
1	5	5	24	**55**

X. How many different paths are there from A to B?



Top layer:

1 T1	1 T2	1 T3
1 T4	2 T5	3 T6
1 T7	3 T8	6 T9

Middle layer:

1 M1	2 M2	3 M3
2 M4	6 M5	12 M6
3 M7	12 M8	30 M9

Bottom layer:

1 B1	3 B2	6 B3
3 B4	12 B5	30 B6
6 B7	30 B8	90 B9

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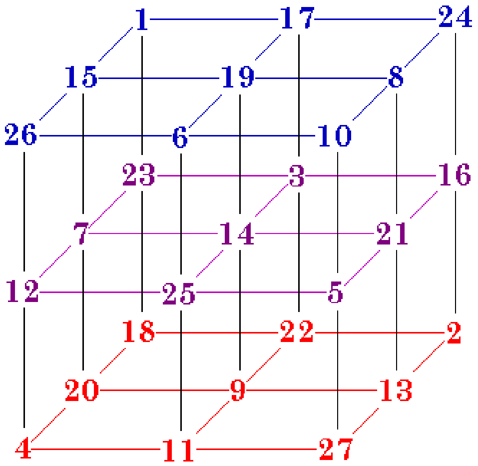
1. Fill in the number of paths for the top layer.
2. From T1 down to M1. From M1 down to B1.
3. M2: From T2 and from M1.
4. M3: From T3 and from M2.
5. M4: From M1 and T4.
6. M5: From M2, M4 and T5.
7. M7 ← M4 + T7
8. M8 ← M7 + M5 + T8
9. M9 ← M6 + M8 + T9
10. B3 ← B2 + M3
11. B7 ← B4 + M7
12. B5 ← B2 + B4 + M5
13. B6 ← B3 + B5 + M6
14. B8 ← B5 + B7 + M8
15. B9 ← B6 + B8 + M9

There are 90 different paths from A to B.

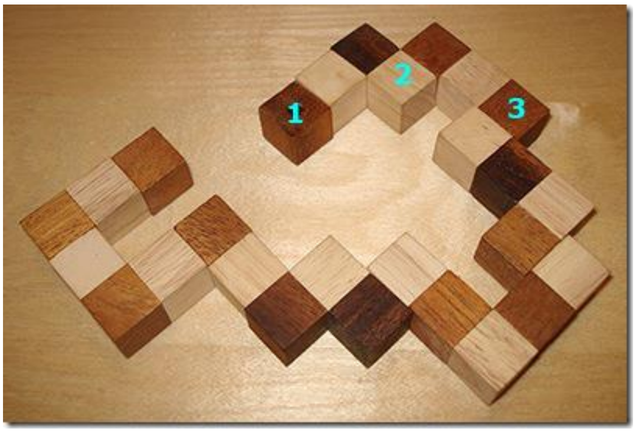
https://www.khanacademy.org/math/math-for-fun-and-glory/puzzles/brain-teasers/v/3-d-path-counting-brain-teaser

X Addisioneel (interessant) / Additional (interesting)

1. Hoeveel verskillende paaie is daar in die *xy*-vlak van (1,5) tot (7,10) as 'n pad uit stappe bestaan van een eenheid na regs of een eenheid op?
2. https://www.youtube.com/watch?v=PwiRVHoA8MA 4:53 "*Mastermind*"
3. 3-D Towervierkant / 3-D Magic Square



4. *Snake* puzzle



WAARSKYNLIKHEID: INLEIDING

PROBABILITY: INTRODUCTION

0 ≤ P(an event will occur) ≤ 1
0 ≤ P(dat 'n gebeurtenis sal plaasvind) ≤ 1

P(an event is sure to occur) = 1
P('n gebeurtenis sal definitief plaasvind) = 1

P(an event will never occur) = 0
P('n gebeurtenis sal nooit plaasvind nie) = 0

P(A) is the probability that event A will take place.
P(A) is die waarskynlikheid dat gebeurtenis A sal plaasvind.

P(A') is the probability that event A will not take place.
A' is the complement of A.
P(A') is die waarskynlikheid dat gebeurtenis A nie sal plaasvind nie.
A' is die komplement van A.

P(A) = 1 – P(A')
P(A') = 1 – P(A)

	Description / Beskrywing
biased coin *sydige muntstuk*	The probability of obtaining a H or a T is not equal. Die waarskynlikheid om kop of stert te kry is nie dieselfde nie.
cardinal number *kardinaalgetal*	n(A) is the number of elements in A. n(A) is die hoeveelheid elemente in A.
complementary events sample set, e.g. {H, T} *komplementêre gebeurtenisse* Steekproefversameling bv. {H, T}	The complementary event A' is the event that contains all the outcomes within the sample set (or sample space) S that are not contained by the event A. Die komplementêre gebeurtenis A' is die gebeurtenis wat al die uitkomste in die steekproefversameling S bevat wat nie in die gebeurtenis A is nie. **Example**: If S = {1; 2; 3; 4; 5} and A = {2; 4} then A' = {1; 3; 5} **Voorbeeld**: As S = {1; 2; 3; 4; 5} en A = {2; 4} dan is A' = {1; 3; 5} One can say that P(A') = P(not A) / Ons kan sê dat P(A') = P(nie A) P(A') = 1 – P(A), P(A) = 1 – P(A') **Example**: Throw two dice simultaneously. The probability that at least one of them is a six is (1 – the probability that none of them is a six). **Voorbeeld**: Rol twee dobbelsteentjies gelyktydig. Die waarskynlikheid dat ten minste een van hulle 'n ses is, is (1 – die waarskynlikheid dat geen hulle 'n ses is nie). $P(\text{6 vir een dobbelsteentjie}) = \frac{1}{6} = P(\text{6 for one die})$ $P(\text{nie 'n 6}) = 1 - \frac{1}{6} = \frac{5}{6} = P(\text{not a 6})$ $P(\text{geen sesse}) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36} = P(\textit{no}\text{ sixes})$ $P(\text{1 of albei is 6}) = 1 - \frac{25}{36} = \frac{11}{36} = P(\text{1 or both are 6})$

	$P(\text{albei is 6}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = P(\text{both are 6})$
conditional probability *voorwaardelike waarskynlikheid*	P(A | B) is the probability of event A occurring given that event B has occurred. P(A | B) is die waarskynlikheid dat gebeurtenis A plaasvind gegee dat gebeurtenis B wel plaasgevind het. ★ With replacement → events always independent. ★ Met terugplasing → gebeurtenisse altyd onafhanklik 🌢 Without replacement → events always dependent. 🌢 Sonder terugplasing → gebeurtenisse altyd afhanklik The following are the same: ① Rolling two dice (or coins) simultaneously. ② Rolling two dice (or coins) successively. Die volgende is dieselfde: ① Gooi twee dobbelsteentjies (of muntstukke) gelyktydig. ② Gooi twee dobbelsteentjies (of muntstukke) na mekaar. **Example**: There are 4 red and 6 blue marbles in a box. 1. Draw a marble, **<u>replace it</u>** (put it back in the box) and make another draw (independent events). 2. Draw a marble, do not replace (do not put it back in the box) it and make another draw (dependent events). **1.** $P(B_1) = \frac{6}{10} = \frac{3}{5}$ is the probability that the first marble taken out is blue. $P(R_1) = \frac{4}{10} = \frac{2}{5}$ is the probability that the first marble taken out is red. $P(R_1 \text{ and } R_1) = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$ is the probability that the first and second marbles are both red; with replacement. $P(B_1 \text{ and } B_2) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$ is the probability that the first and second marbles are both blue; with replacement. $P(R_1 \text{ and } B_2) = \frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$ is the probability that the first one is red and the second marble is blue; with replacement. $P(B_1 \text{ and } R_2) = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$ is the probability that the first one is blue and second marble is red; with replacement. **2.** $P(B_1) = \frac{6}{10} = \frac{3}{5}$ is the probability that the first marble taken out is blue and it is not replaced. Now there are only 9 marbles left in the box; 5 blue ones and 4 red ones. $P(B_2) = \frac{5}{9}$ is the probability that the second marble taken out is also blue. $P(B_1)P(B_2 \mid B_1) = \frac{5}{9} \times \frac{6}{10} = \frac{30}{90} = \frac{1}{3}$ $P(R_1)P(R_2 \mid R_1) = \frac{3}{9} \times \frac{4}{10} = \frac{2}{15}$ $P(B_1)P(R_2 \mid B_1) = \frac{4}{9} \times \frac{6}{10} = \frac{4}{15}$

	$P(R_1)P(B_2 \mid R_1) = \frac{6}{9} \times \frac{4}{10} = \frac{4}{15}$ Die notasie word na die volgende voorbeeld verduidelik. **Voorbeeld** In baie gevalle kan 'n mens 'n beter benadering vir die waarskynlikheid van 'n gebeurtenis kry soos wat meer inligting bekend word. As jy byvoorbeeld elke Vrydag na dieselfde kafee toe gaan om middagete te koop dan is die waarskynlikheid dat jy binne 15 minute jou bestelling sal hê 0.9. As jy 'n halfuur later as gewoonlik daar aankom, is die waarskynlikheid om binne 15 minute jou bestelling te kry egter 0.7. Dit is ook 'n voorbeeld van voorwaardelike waarskynlikheid. **Example** In many situations, as more information becomes available, we are able to revise our estimates for the probability of further outcomes or events happening. For example, suppose you go out for lunch at the same place every Friday and you are served lunch within 15 minutes with probability 0.9. If you arrive 30 minutes later than usual the probability of being served within 15 minutes is 0.7. This is an example of conditional probability. Die notasie vir voorwaardelike waarskynlikheid "gebeurtenis A vind plaas slegs as gebeurtenis B plaasgevind het " is "A | B" (A gegee B). Die simbool | is 'n vertikale strepie en dit beteken nie deling nie. P(A | B) dui aan dat gebeurtenis A sal plaasvind slegs as gebeurtenis B reeds plaasgevind het. Dit word ook geskryf as $P(A\mid B) = \frac{P(A \cap B)}{P(B)}$. $P(A \cap B)$ is die onvoorwaardelike waarskynlikheid dat beide gebeurtenisse A en B plaasvind. $P(B)$ is die onvoorwaardelike waarskynlikheid dat gebeurtenis B plaasvind. The notation for conditional probability "event A occurs given that event B has occurred" is "A | B" (A given B). The symbol | is a vertical line and does not imply division. P(A | B) denotes the probability that event A will occur given that event B has occurred already. This is also written as $P(A\mid B) = \frac{P(A \cap B)}{P(B)}$. $P(A \cap B)$ is the unconditional probability that both events A and B occur. $P(B)$ is the unconditional probability that event B occurs. Twee gebeurtenisse, A en B, is onafhanklik as P(A) = P(A | B) en P(B) = P(B | A). Two events, A and B, are independent if P(A) = P(A | B) and P(B) = P(B | A).
Coin Muntstuk	$P(H) = \frac{1}{2}$ $P(T) = \frac{1}{2}$ As 'n mens twee muntstukke gelyktydig gooi, is daar vier moontlikhede: HH, HT, TH en TT. P(HH) = $\frac{1}{4}$ P(een H) = $\frac{1}{2}$ P(drie H) = 0 P(ten minste een H) = $\frac{3}{4}$

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	P(H en T) = 0